On the relationship between finger width, velocity, and fluxes in thermohaline convection

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Double-diffusive finger convection occurs in many natural processes. The theories for double-diffusive phenomena that exist at present consider systems with linear stratification in temperature and salinity. The double-diffusive systems with step change in salinity and temperature are, however, not amenable to simple stability analysis. Hence factors that control the width of the finger, velocity, and fluxes in systems that have step change in temperature and salinity have not been understood so far. In this paper we provide new physical insight regarding factors that influence finger convection in two-layer double-diffusive system through two-dimensional numerical simulations. Simulations have been carried out for density stability ratios ($R_f$) from 1.5 to 10. For each density stability ratio, the thermal Rayleigh number ($Ra_T$) has been systematically varied from $7 \times 10^5$ to $7 \times 10^8$. Results from these simulations show how finger width, velocity, and flux ratios in finger convection are interrelated and the influence of governing parameters such as density stability ratio and the thermal Rayleigh number. The width of the incipient fingers at the time of onset of instability has been shown to vary as $Ra_T^{-1/3}$. Velocity in the finger varies as $Ra_T^{1/3}/R_p$.

Results from simulation agree with the scale analysis presented in the paper. Our results demonstrate that wide fingers have lower velocities and flux ratios compared to those in narrow fingers. This result contradicts present notions about the relation between finger width and flux ratio. A counterflow heat-exchanger analogy is used in understanding the dependence of flux ratio on finger width and velocity. © 2009 American Institute of Physics. [DOI: 10.1063/1.3070527]

I. INTRODUCTION

Double-diffusive finger convection is important in many engineering applications and natural processes. For example double-diffusive finger convection plays an important role in the transport of salt, heat, and in upwelling of nutrients in oceans, in microstructure formation in the solidification of alloys, and in determining magma characteristics in magma chambers. In these examples, different aspects of alloys, and in determining magma characteristics in oceans, in microstructure formation in the solidification of the transport of salt, heat, and in upwelling of nutrients in double-diffusive finger convection plays an important role in engineering applications and natural processes. In this paper we provide new physical insight regarding factors that influence finger convection in two-layer double-diffusive system through two-dimensional numerical simulations. Simulations have been carried out for density stability ratios ($R_f$) from 1.5 to 10. For each density stability ratio, the thermal Rayleigh number ($Ra_T$) has been systematically varied from $7 \times 10^5$ to $7 \times 10^8$. Results from these simulations show how finger width, velocity, and flux ratios in finger convection are interrelated and the influence of governing parameters such as density stability ratio and the thermal Rayleigh number. The width of the incipient fingers at the time of onset of instability has been shown to vary as $Ra_T^{-1/3}$. Velocity in the finger varies as $Ra_T^{1/3}/R_p$.

Results from simulation agree with the scale analysis presented in the paper. Our results demonstrate that wide fingers have lower velocities and flux ratios compared to those in narrow fingers. This result contradicts present notions about the relation between finger width and flux ratio. A counterflow heat-exchanger analogy is used in understanding the dependence of flux ratio on finger width and velocity. © 2009 American Institute of Physics. [DOI: 10.1063/1.3070527]

In laboratory experiments and observations in the ocean, salt fingers sandwiched between well-mixed layers have been observed. The mixed layer height in laboratory experiments is of the order of 10 cm while in the ocean it varies from a fraction of a meter to tens of meters [see Refs. 8 and 10 and Fig. 1(a)]. Vertical salinity and temperature traces obtained from oceans indicate the presence of many layers of different depths having different salinities and temperature contrasts [Fig. 1(a)]. Transport of salt and heat occurs through these layered structures. The rates of transport of two components depend on the characteristics of the convection cell, such as fluid velocity and the width of the fingers, and the intensity of turbulence in the mixed layers across the finger region. Parameterization of fluxes across these layers is based on salinity and temperature contrast between the two adjacent layers and has been more successful in the case of diffusive regime than in the finger regime.

The following approaches have been used to predict the rate of transport of salt and heat and the characteristics of finger convection. Analytical prediction of the fluxes is either based on the fastest growing mode of the finger or based on maximizing the buoyancy transport. There are many two-layer laboratory experiments and field observations reporting individual fluxes of heat and salt and heat/salt flux ratio ($R_f$) [e.g., Refs. 23 and 24]. Flux ratio $R_f$ is the ratio between the fluxes of a density anomaly due to heat and that due to salt as given by

$$R_f = \frac{\text{Heat flux}}{\text{Salt flux}}.$$
where \( F_T \) and \( F_S \) are the interfacial convective fluxes of temperature and salinity respectively, \( W' \), \( T' \), and \( S' \) are the dimensional fluctuating vertical velocity, temperature, and salinity respectively. Constants \( \beta_T \) and \( \beta_S \) are the expansion coefficients of temperature and salinity, respectively, and \( \langle \cdot \rangle \) indicates an average of a quantity over time and horizontal direction. Investigators often correlate the flux ratio with a single governing parameter called the density stability ratio \( R_p \) defined as

\[
R_p = \frac{\beta_T \Delta T}{\beta_S \Delta S} \geq 1, \tag{1.2}
\]

where \( \Delta T \) and \( \Delta S \) are the temperature and salinity differences across the fingering interface [refer to Fig. 1(b)]. The density stability ratio \( R_p \) is the ratio between the thermal Rayleigh number \( R_T \) and the salinity Rayleigh number \( R_S \), where \( R_T \) and \( R_S \) are defined as

\[
R_T = \frac{\beta_T \Delta T H^3}{v k_T}, \quad R_S = \frac{\beta_S \Delta S H^3}{v k_T}, \tag{1.3}
\]

where \( g \) is the acceleration due to gravity, \( v \) is the kinematic viscosity of the fluid, \( k_T \) is the thermal diffusivity, and \( H \) is the height of the computational domain [Fig. 1(b)]. Figure 2 shows the values of \( R_f \) reported by the various investigators. The value of \( R_f \) obtained from the heat-salt laboratory experiments \(^{18,20,22} \) is around 0.60–0.65 and is consistent with the prediction of the flux ratios of the fastest growing fingers. The experimental data from McDougall and Taylor,\(^ {21} \) which are in the lower range of \( R_p \), indicate much lower values of flux ratios (0.3–0.5). The lower value of \( R_f \) observed by them is still not understood.\(^ {14,26} \) Taylor and Bucens\(^ {22} \) attributed the low value of \( R_f \) to a possible contribution from steady fingers or fingers of maximum buoyancy flux. Schmitt\(^ {27} \) attributed the low values of \( R_f \) obtained by MacDougall and Taylor to a different stability criterion that is applicable to the fingers at low-density ratios. Linden\(^ {19} \) observed a value of \( R_f \) around 0.1 in the heat-sugar double-diffusive experiments. The large range of variation in flux ratio (\( R_f = 0.10–0.70 \)) is much higher than the uncertainties in measurement.

Many investigators, for example, Piacsek and Toomre\(^ {28} \), Shen\(^ {29,30} \) and Shen and Veronis\(^ {31} \) reported numerical simulations of salt fingers for a heat-salt system. The values of \( R_f \) obtained from these simulations also have scatter (Fig. 2). Piacsek and Toomre\(^ {28} \) found a linear dependence of \( R_f \) on the

![FIG. 1. (Color online) (a) Ocean temperature and salinity profiles indicating the presence of staircase structure with large variations in layer depths (Ref. 10). (b) Schematic indicating one of the step changes (idealized) in the staircase structure used in the present study. The diagram also indicates initial profiles and boundary conditions used in simulations. In the diagram, \( S, T, \) and \( \rho \) represent salinity, temperature, and density, respectively. Subscripts \( T \) and \( b \) represent corresponding values for the top and bottom layers, respectively. \( T_f = T_b + \Delta T, S_f = S_b + \Delta S, \) and \( R_f = \beta_T \Delta T / \beta_S \Delta S \) is the density stability ratio. \( H \) and \( b \) represent dimensional height and width, whereas \( \Delta z \) is the nondimensional height of the computational domain. \( u \) and \( w \) are the nondimensional horizontal and vertical velocities.](image1)

![FIG. 2. (Color online) Variation of flux ratio \( (R_f) \) as a function of density stability ratio \( (R_p) \) reported by various investigators. Functional relationships for \( R_f \) with \( R_p \) given by Kelly (Ref. 38) \( [R_f = 0.35 + 0.62 e^{0.66 R_p - 1}] \), Kunze (Ref. 14) \( [R_f = 0.44 R_p (R_p - 1)] \), and Boyd and Perkins (Ref. 39) \( [R_f = 0.44 / (R_p - 1)] \) show a similar trend, i.e., \( R_f \approx 1 \) when \( R_p \approx 1 \). However, the variation increases at higher values of \( R_p \). Kunze’s theory of fastest growing fingers captures the observations from the laboratory and numerical experiments except for that reported by McDougall and Taylor (Ref. 21). McDougall and Taylor reported much lower values of \( R_f \) which are not in agreement with the previous observations. The horizontal dotted line predicts a constant value of \( R_f \) based on maximum buoyancy flux principle.](image2)
density stability ratio $R_f$, which is not in agreement with the observations in the laboratory experiments and theory. Shen\textsuperscript{30} obtained a value of $R_f=0.50$ for density stability ratio in the range of $2 < R_f < 4$. Shen and Veronis\textsuperscript{31} argued that $R_f$ for the heat-salt system should depend on the values of the thermal and salinity Rayleigh numbers. They, however, did not discuss the nature of this dependence. The ratio of diffusivity of salt to heat is called diffusivity ratio ($\tau$). In some of the recent numerical simulations of two dimensional (2D) and three dimensional (3D) salt-finger\textsuperscript{32-37} simulations are carried out at a high value of diffusivity ratio ($\tau$) and at a low value of the Prandtl number and then the results are extrapolated to the lower value of $\tau$, relevant to heat-salt system. These results support the observations from the laboratory experiments. One of the important observations made by most authors is that $R_f \rightarrow 1$ when $R_\rho \rightarrow 1$.

There also exist significant variations in the prediction of flux ratio obtained through $R_f R_\rho$ correlations reported in the literature. Some of the correlations of $R_f$ with $R_\rho$ are discussed below. Kelly\textsuperscript{38} empirically fitted the available data of $R_f$ variation with $R_\rho$ (Fig. 2). The correlation proposed by Kunze\textsuperscript{14} is based on the theory of fastest growing finger mode, and that by Boyd and Perkins\textsuperscript{39} is the empirical fit; all these correlations are shown in Fig. 2. It is evident from the figure that these correlations capture the trend of $R_f \rightarrow 1$ when $R_\rho \rightarrow 1$, a general trend that has been observed in laboratory experiments and numerical simulations. Among the reported correlations, Kunze’s correlation is the best and it also captures the values of $R_f$ obtained in most of the laboratory experiments except for the values reported by McDougall and Taylor.\textsuperscript{21} On the other hand, correlation based on the assumption that maximum buoyancy flux is transported by the finger structures predicts a constant value of $R_f=0.25$ (e.g., Refs. 16 and 17), which is far lower than most of the observations and is independent of $Ra_F$ and $R_\rho$.

Another interesting issue is the relation between flux ratio $R_f$ and other system parameters such as finger width $\lambda$ and velocity. The prevalent notion is that wide fingers have higher flux ratio compared to narrow fingers. This is reflected in the argument by Schmitt\textsuperscript{15} that a very wide finger can be expected to dissipate less energy than a thin finger. Similarly, a thin finger will be highly dissipative and will result in a low value of $R_f$. Howard and Veronis\textsuperscript{47} also argued that salt flux achieves its maximum for infinitesimally thin fingers because the horizontally averaged vertical velocity is at its maximum there. Heat flux goes to zero for very thin fingers. This again implies that wide fingers have high $R_f$ compared to narrow fingers, which can be seen in Fig. 3 of their paper. They obtained a value of $R_f=0.25$, when the fingers transport maximum buoyancy flux. Schmitt\textsuperscript{15} and Eisenman\textsuperscript{40} made similar observations about wide fingers with high values of $R_f$.

Based on the above discussion we see that several issues are still unresolved in double-diffusive finger convection. If the flux ratio $R_f$ is a function of $R_\rho$, alone, then a large scatter in Fig. 2 is not easy to explain. Interdependence between finger width and velocity and how they affect flux ratio have not been demonstrated. As the double-diffusive convection is a two-parameter problem ($Ra_F$ and $Ra_\rho$), it is natural to expect $R_f$ dependence on $Ra_F$ and one of the Rayleigh numbers. Even though some researchers highlighted $R_f$ dependence on two parameters (for example, Shen and Veronis\textsuperscript{31}), there are no correlations that relate flux ratio on parameters such as $Ra_F$, finger width, and velocity. Hence this issue demands further investigation.

In this paper, we have made a systematic attempt to address these issues. We present results from numerical simulations and scaling analysis, which elucidate the relation between finger width, finger velocity, and other governing parameters with the flux ratio. Two-dimensional (2D) numerical simulations are performed for a heat and salt system with $R_\rho$, equal to 1.5, 2, 6, and 10. For each value of $R_\rho$, we have systematically varied $Ra_F$ from $7 \times 10^3$ to $7 \times 10^8$ (see Table I). Thus, for a given Rayleigh number, we have results for four density stability ratios and for a given density stability ratio we have results for six Rayleigh numbers.

Our 2D simulations can be justified by the recent observations of Kamakura and Ozoe,\textsuperscript{11} who showed that characteristics of finger convection are essentially the same for both in 3D and 2D numerical simulations. The fluxes obtained from 2D simulations are similar to measurements.\textsuperscript{29,26} The secondary instabilities that disrupt growing fingers are similar in two and three dimensions.\textsuperscript{43} Finally, since the Reynolds numbers for salt-finger convection are low (<10), the turbulent dynamics of 3D flows do not have a dominant influence.\textsuperscript{44} However, one has to be careful while comparing results from the 2D simulations with the results from field observations or the experiments in a Hele-Shaw cell because the present simulation does not account for wind shear and other factors present in the field or nonlinear interactions from the third dimension and drag exerted by the walls of the Hele-Shaw cell on the fluid.

The paper is organized as follows. In Sec. II, we present the basic equations governing salt-finger convection and the

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<th>$Ra_F$</th>
<th>$7.0 \times 10^3$</th>
<th>$7.0 \times 10^4$</th>
<th>$7.0 \times 10^5$</th>
<th>$7.0 \times 10^6$</th>
<th>$7.0 \times 10^7$</th>
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<tbody>
<tr>
<td>$Ra_{\rho 1}$</td>
<td>$4.667 \times 10^3$</td>
<td>$4.667 \times 10^4$</td>
<td>$4.667 \times 10^5$</td>
<td>$4.667 \times 10^6$</td>
<td>$4.667 \times 10^7$</td>
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<tr>
<td>$Ra_{\rho 2}$</td>
<td>$3.5 \times 10^3$</td>
<td>$3.5 \times 10^4$</td>
<td>$3.5 \times 10^5$</td>
<td>$3.5 \times 10^6$</td>
<td>$3.5 \times 10^7$</td>
</tr>
<tr>
<td>$Ra_{\rho 3}$</td>
<td>$1.167 \times 10^3$</td>
<td>$1.167 \times 10^4$</td>
<td>$1.167 \times 10^5$</td>
<td>$1.167 \times 10^6$</td>
<td>$1.167 \times 10^7$</td>
</tr>
<tr>
<td>$Ra_{\rho 4}$</td>
<td>$7.0 \times 10^3$</td>
<td>$7.0 \times 10^4$</td>
<td>$7.0 \times 10^5$</td>
<td>$7.0 \times 10^6$</td>
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governing parameters. Results from the numerical simulations of the equation are described in Sec. III followed by conclusions in Sec. IV.

II. NUMERICAL MODELING OF SALT FINGERS

Numerical simulations have been carried out for understanding the role of different parameters that govern the vertical transport of fluxes in double-diffusive convection. We have used Boussinesq and dilute solution approximations. The fluid density is a function of concentration and temperature and is expressed in the linear form as 

\[ \rho = \rho_0 \left[ 1 - \beta_T (T - T_0) + \beta_S (S - S_0) \right] \]

where \( \rho_0 \) is the reference density. The 2D, unsteady equations that govern the double-diffusive systems are continuity equation (2.1), momentum equations in \( x \) [Eq. (2.2)] and \( z \) [Eq. (2.3)] directions, and heat [Eq. (2.4)], and salt concentration [Eq. (2.5)] conservation equations, which are presented below in the nondimensional form:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0 \]

\[ \frac{\partial u}{\partial t^*} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = - \frac{\partial p}{\partial x} + Pr \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) \]

\[ \frac{\partial v}{\partial t^*} + u \frac{\partial v}{\partial x} + w \frac{\partial v}{\partial z} = - \frac{\partial p}{\partial z} + Pr \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) - Ra_S Pr S^* + Ra_T Pr T^* \]

\[ \frac{\partial T^*}{\partial t^*} + u \frac{\partial T^*}{\partial x} + w \frac{\partial T^*}{\partial z} = \left( \frac{\partial^2 T^*}{\partial x^2} + \frac{\partial^2 T^*}{\partial z^2} \right) \]

\[ \frac{\partial S^*}{\partial t^*} + u \frac{\partial S^*}{\partial x} + w \frac{\partial S^*}{\partial z} = \frac{Pr}{Sc} \left( \frac{\partial^2 S^*}{\partial x^2} + \frac{\partial^2 S^*}{\partial z^2} \right) \]

The Prandtl number (Pr) and Schmidt number (Sc) are defined as follows:

\[ Pr = \frac{v}{k_T}, \quad Sc = \frac{v}{k_S} \]

In the present study, values of \( (Pr, Sc) = (7,700) \) have been kept constant in all simulations, which corresponds to the aqueous-thermohaline system. The scaling parameters used to develop the nondimensional governing equations (2.1)–(2.5) are \( x=X/H, \quad z=Z/H, \quad u=U/(k_T/H), \quad w=W/(k_f/H), \quad T^*=(T-T_b)/\Delta T, \quad S^*=(S-S_b)/\Delta S, \quad p=P/(k_T \rho_0/H^2), \) and \( t^*=t/(H^2/k_T) \) for length scales in the horizontal and vertical directions, corresponding velocity components, temperature, species concentration, pressure, and time, respectively. Since we have chosen \( k_T \) (thermal diffusivity) for scaling time and velocities, the Peclet number (Pe) does not appear in the governing equations. This scaling leads to the appearance of \( k_T \) in the definitions of both salinity and thermal Rayleigh numbers and is a normal practice in the literature on double-diffusive convection. Here \( T_b \) and \( S_b \) are the bottom layer temperature and salinity, \( k_S \) is the molecular diffusivity of the salt. \( \Delta T \) and \( \Delta S \) are the temperature and concentration differences between the top and bottom layers [Fig. 1(b)]. The variation of \( T^* \) and \( S^* \) would be between \([0, 1] \). The ratio of the two Rayleigh numbers gives the density stability or buoyancy ratio, \( R_T = Ra_T/Ra_S = \beta_T \Delta T/\beta_S \Delta S \). Most of the results presented here are in the nondimensional form; however, in a few cases, for example, the physical time presented in Fig. 3, we have presented dimensional values. For obtaining dimensional values presented in the paper, the following physical properties are used: thermal diffusivity \( k_f = 1.4 \times 10^{-5} \text{ m}^2/\text{s} \), salinity diffusivity \( k_S = 1.4 \times 10^{-9} \text{ m}^2/\text{s} \), kinematic viscosity \( \nu = 1 \times 10^{-6} \text{ m}^2/\text{s} \), salinity expansion coefficient \( \beta_S = 8 \times 10^{-4}/\text{S} \), and thermal expansion coefficient \( \beta_T = 2 \times 10^{-4}/\text{C} \), and the computation domain height is \( H = 0.15 \text{ m} \).

Hot and salty fluid flows over cold and fresh water in a two-layer system [Fig. 1(b)]. All the four walls of the domain were assumed to be adiabatic with no mass flux across the boundary. Slip conditions were used for the vertical velocity on the sidewalls. A step profile for temperature and concentration was used as the initial condition. We preferred an initial step profile over initial linear profile since most of the laboratory experiments, in which fluxes have been measured have an initial two-layer configuration. Equations (2.1)–(2.5) were solved numerically using finite volume formulation. The simulation was performed on a COMPAQ Alpha-server ES40 having four central processing units per server with 667 MHz processor speed. Solutions were obtained by marching in time using SIMPLER algorithm along with alternate direction implicit method. The numerical method adopted in this paper is preferred over the spectral methods (e.g., Galerkin truncation with integrating factors) since the discontinuities at the interface in the two-layer system can be handled well with the numerical scheme used here. The code has been validated by reproducing results published in the literature for single component convection and double-diffusive convection (Refs. 47 and 48, both in the diffusion regime). As will be shown in the later part of this paper, \( R_f \) values in the finger regime from the present simulations compare well with observations.

The sensitivity of the results to the number of nodes chosen was examined at the highest and lowest Rayleigh numbers at fixed \( R_f \). The effects of different node sizes and time steps were studied on finger width, maximum kinetic energy \( (E_k) \), and flux ratio \( R_f \). The effect on finger structure was investigated by observing the variation in the number of fingers \( (N_f) \) in the computational domain. The number of nodes at the highest \( Ra_T \) was \((N_x, N_z)=(1200, 400)\) for an aspect ratio \((B/H)\) of 1, whereas at the lowest \( Ra_T \), they were \((N_x, N_z)=(600, 200)\) for the same aspect ratio. For \( Ra_T=7 \times 10^8 \), the flux ratio increased from 0.689 to 0.728 when the number of nodes was increased from 251 × 251 to 401 × 401. The flux ratio in the simulations increases by about 5% when the number of nodes was increased from 251 × 251 to 301 × 301, and further increasing the number of nodes to 401 × 401 resulted in only 0.4% increase in the flux ratio. This variation is less than the errors in the measurement of flux ratio. More details on validation of the code and accuracy of the simulations can be found in Refs. 49–51.

Different numbers of nodes and time steps were used...
depending on the Rayleigh number and aspect ratio of the computational domain. Typically, a nondimensional time step corresponding to a physical time step of $\Delta t = 0.005$ s was used at high $Ra_T$ and $\Delta t = 0.05$ s was used at low $Ra_T$. Simulations were carried out with different aspect ratios ($B/H$) varying from 0.333 (for thin fingers) to 2 (for thick fingers) so as to accommodate at least a few fingers in the computational domain. $B$ and $H$ are the dimensional width and height of the computational domain, respectively.

III. RESULTS AND DISCUSSIONS

The results from numerical simulations of double-diffusive finger system for the various values of $R_{p}$ and $Ra_T$ are presented in this section. In Fig. 3, the evolutions of temperature ($T^*$) and salinity ($S^*$) fields at an instant for density ratio $R_{p}=1.5$ have been presented for thermal Rayleigh numbers $Ra_T=7 \times 10^6$ [Figs. 3(a) and 3(b)] and $Ra_T=7 \times 10^4$ [Figs. 3(c) and 3(d)]. A regular array of thin fingers evolves at the interface within a few minutes from the start of the simulation when the $Ra_T$ is $7 \times 10^6$. Temperature ($T^*$) and salinity ($S^*$) fields look similar with $T^*$ field showing little effect of lateral diffusion of heat. However, when $Ra_T$ is reduced from $7 \times 10^6$ to $7 \times 10^4$, holding $R_{p}$ at the same value, the finger width increases. At low $Ra_T$, the difference between salinity ($S^*$) and temperature ($T^*$) fields becomes more apparent [Figs. 3(c) and 3(d)]. With decreasing $Ra_T$, the finger structure changes from a convection-dominated structure to the one in which thermal diffusion dominates. As $Ra_T$ decreases, the time required for evolution of the same length of finger increases tenfold. Convective structures for $R_{p}=2$, 6, and 10 indicate a similar trend. Lateral diffusion of heat is more pronounced at low $Ra_T$ and at high $R_{p}$. Another important aspect to note is that the lateral diffusion of heat is more pronounced when fingers are wider rather than when they are narrower, which is contrary to the assumption made by earlier investigators.

Figure 4 demonstrates the variation of fluctuating salinity, temperature, and vertical velocity ($S''$, $T''$, and $w''$) for high and low thermal Rayleigh numbers and at the same value of density stability ratio $R_{p}=1.5$. Fluctuating quantities are plotted at the midsthight of the computational domain and in the regions between $1 < x < 2$ of Fig. 3. We notice that the magnitudes of $S''$ vary between $\pm 0.5$ [Fig. 4(a)] for both narrow and wide fingers. This means that the salinity is virtually retained within the fingers, and the horizontal diffusion of salinity is negligible for both narrow and wide fingers. On the other hand, the magnitudes of $T''$ and $w''$ in wide fingers are significantly lower compared to those in narrow fingers [Figs. 4(b) and 4(c)]. In Fig. 4(d), we present variation of vertical heat flux with the finger width for various values of density stability ratios and thermal Rayleigh numbers. Vertical heat flux transported by wide fingers is low in comparison to that by narrow fingers. This set of results indicates that in wider fingers most of the thermal anomalies are short-circuited in the lateral direction and little is advected in the vertical direction. This result is contrary to the assumption.
(for example, Ref. 13) that wider fingers maintain larger $\beta_T T'$ and have higher flux ratio compared to narrow fingers.

In Fig. 5 we present variation of the convective heat flux $\beta_T W' T''$, m/s with salinity flux $\beta_S W' S''$, m/s, which indicates how the variations in Rayleigh number and density stability ratio control individual fluxes and flux ratio. The dashed line in the figure indicates the condition when the heat flux is equal to the salinity flux or, for the points on the dashed line, the flux ratio will be unity. All data points lie above the dashed line, which indicates $R_f$ ($\beta_T W' T'' / \beta_S W' S''$) is less than unity for all of them. When the Rayleigh number is high and the density stability ratio is low (upper right corner of the plot) data points asymptotically approach the dashed line ($R_f$ approaches unity). In the other limit with decreasing Rayleigh number and increasing stability ratio (towards the lower left corner of the plot) data points move up and away from the dashed line ($R_f$ decreases from unity). The reason for these variations will be elucidated later through a counterflow heat exchanger model.

The following results show how governing parameters $R_f$ and $R_T$ control the finger width. In Fig. 6, we present salinity fields for a set of six simulations in which $R_T$ was systematically decreased from $7 \times 10^3$ to $7 \times 10^5$ while holding the value of density stability ratio at 6 ($R_p$=6). For this set of simulations, presented in Fig. 6, the aspect ratio of the computational domain was 1.5 for all six cases, which highlights variation of the finger width. However, it is evident in Fig. 6(a) that the number of fingers is so large that they are poorly resolved. Hence, as stated in the earlier part of the paper, for high Rayleigh number simulations, a smaller aspect ratio computation domain with a higher number of nodes is used. The finger width shows considerable variation.
At high $Ra_T = 7 \times 10^8$ [Fig. 6(a)], very thin fingers form and the number of fingers is about 125. Finger thickness increases as we go to lower Rayleigh numbers. At the lowest Rayleigh number ($Ra_T = 7 \times 10^3$), Fig. 6(f), we observe only three fingers (two in the center and two halves at the sides) within the computational domain. This trend in variation of finger thickness as a function of Rayleigh numbers is consistent with the experimental observation of Pringle and Glass. One has to be cautious in extending the comparison of the present 2D simulation to the results observed in a Hele-Shaw cell as the present simulation does not account for 3D effects and the drag exerted by Hele-Shaw cell boundaries on the fluid. Note that as the Rayleigh number decreases from $7 \times 10^8$ to $7 \times 10^3$, the physical time required in our simulations for evolving similar length fingers increases from about 3 min to about 14 h.

Horizontally averaged vertical velocity at the midheight of the computation domain was stored during the simulation. The vertical velocity attains a maximum value just before the fingers reach the top and bottom boundaries. We have considered this maximum value in the vertical velocity as the velocity scale. Values obtained in the simulation for each parameter set is given in Table II. From the salinity fields, similar to the one presented in Fig. 6, the nondimensional width ($\lambda/H$) of the finger at the onset of instability is extracted for all the cases considered in the present study. In Fig. 7 we present the variation of vertical velocity scale with

**FIG. 5.** Variation of convective heat flux ($\beta_p W T^*$, m/s) with salinity flux ($W S^*$, m/s) at the interface for various values of density stability ratios ($R_s$) and thermal Rayleigh numbers ($Ra_T$) are presented. Dashed line indicates the condition when both salinity and heat fluxes are equal. The arrow indicates the direction in which $Ra_T$ is decreasing.

**FIG. 6.** (Color online) Evolution of salinity fields indicating the formation of fingers after the onset of instability at the initial interface for various Rayleigh numbers $Ra_T$ at a fixed $R_s = 6$. The aspect ratio of the computational domain is the same in all the cases. Rayleigh numbers and elapsed time of convection are (a) $Ra_T = 7 \times 10^8$, $t = 2$ min, 45 s, (b) $Ra_T = 7 \times 10^7$, $t = 10$ min, (c) $Ra_T = 7 \times 10^6$, $t = 30$ min, (d) $Ra_T = 7 \times 10^5$, $t = 1$ h, 30 min, (e) $Ra_T = 7 \times 10^4$, $t = 4$ h, 33 min, and (f) $Ra_T = 7 \times 10^3$, $t = 13$ h, 40 min. Note that the effect of molecular diffusion at the interface increases as $Ra_T$ decreases. The nondimensional widths of the finger ($\lambda/H$) are extracted from these salinity fields.
the width of the finger. This plot indicates that narrow fingers have higher velocity in comparison to wide fingers. The finger width is a relatively weak function of $R_p/H^{9267}$ and is inversely proportional to $Ra_T$. The vertical velocity is, however, directly proportional to $Ra_T$ and inversely proportional to $R_p/H^{9267}$.

Pringle and Glass reported similar variation of velocity with Rayleigh number. However, this result contradicts some of the theories, which propose that wide fingers have less dissipation and hence will have higher velocity. We consider next how vertical velocity, finger width, and fluxes and their ratios are interrelated. In the following subsection we present a scaling argument to relate the finger width and velocity with governing parameters $Ra_T$ and $R_p$.

A. Scale analysis

1. Finger width

A double-diffusive system in the finger regime (DDF) with step change across the interface will start with a layer of hot saline water over a layer of cold fresh water. Unlike the double-diffusive system having linear stratification of temperature and salinity, an analytical solution is not available for a DDF with step change in salinity and temperature across the interface ($z=0$). There is no explicit length scale in this problem; both length and time scales evolve through diffusion. One approach could be to use instantaneous temperature and salinity gradients at the interface to estimate the finger width from linear stability analysis. In Fig. 8 schematics of the density anomaly profiles (with respect to the bottom layer) just before the onset of the finger instability are presented. Note that at this time, the net density profile has an unstably stratified region between $z=\pm h/2$. Thus the finger-width estimation based on linear stability analysis of double diffusive system having linear temperature and salinity profiles with stable density stratification cannot be extended to the present situation. As will be shown in the later part of this section, the strength, sign, and extent of the unstable density profile at the interface ($z=\pm h/2$) depends on $Ra_T$, density stability ratio ($R_p/H^{9267}$), and diffusivity ratio ($T$).

However, when the density profile has an unstable section and is of the form of stretched “Z,” the stability condition is similar to the onset of penetrative convection as shown by Matthews. Matthews showed that the onset of convection occurs when the Rayleigh number for the unstable layer exceeds a critical value, and the convection cells form with a wavelength proportional to the thickness of the unstable layer ($\sim 2h$). For the double-diffusive system we can say that

![FIG. 7. Variation of nondimensional finger width and vertical velocity with the governing parameters $Ra_T$ and $R_p$ varied. Velocity is inversely related to finger width. Arrow indicates the direction in which $Ra_T$ is increasing.](image)

TABLE II. Nondimensional vertical velocity scale observed in the present simulations.

<table>
<thead>
<tr>
<th>$R_p$</th>
<th>$7.0 \times 10^8$</th>
<th>$7.0 \times 10^7$</th>
<th>$7.0 \times 10^6$</th>
<th>$7.0 \times 10^5$</th>
<th>$7.0 \times 10^4$</th>
<th>$7.0 \times 10^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>498.22</td>
<td>365.52</td>
<td>149.52</td>
<td>66.47</td>
<td>41.64</td>
<td>14.27</td>
</tr>
<tr>
<td>2.0</td>
<td>430.11</td>
<td>230.84</td>
<td>128.23</td>
<td>53.31</td>
<td>30.41</td>
<td>11.32</td>
</tr>
<tr>
<td>6.0</td>
<td>180.71</td>
<td>80.35</td>
<td>43.13</td>
<td>21.23</td>
<td>13.37</td>
<td>4.25</td>
</tr>
<tr>
<td>10</td>
<td>118.82</td>
<td>53.17</td>
<td>27.09</td>
<td>13.14</td>
<td>10.18</td>
<td>2.56</td>
</tr>
</tbody>
</table>

![FIG. 8. (Color online) Density anomaly plots in the two-layer double diffusive system just before the onset of double diffusive instability. Lines indicated by $\rho_T$ and $\rho_S$ show the contributions of temperature and salinity on the density anomaly and the line indicated by $\rho_b$ shows the combined effect, resulting in the net density anomaly profile. $L_T$ and $L_S$ are the thermal and salinity diffusion lengths, $H$ is the total height of the system, and $h$ is the height of unstable layer at the interface.](image)
convection onset occurs when $Ra_{\text{effective}} \sim Ra_{cr}$, where $Ra_{\text{effective}}$ is the effective Rayleigh number at the interface,

$$Ra_{\text{Ref}0} = \frac{Ra_{\text{Ref}}}{\tau} \approx Ra_{cr}.$$ 

Here $Ra_{\text{Ref}0}$ and $Ra_{\text{Ref}0}$ are the instantaneous-gradient thermal and salinity Rayleigh numbers at the interface, which are defined as follows:

$$Ra_{\text{Ref}0} = \frac{g \beta_T T_{Z0} h^4}{v k_T} = \frac{g \beta_T \Delta T h^4}{2L_T v k_T} = \frac{g \beta_T \Delta T h^3}{2L_T}$$

and

$$Ra_{s0} = \frac{g \beta_s S_{Z0} h^4}{v k_T} = \frac{g \beta_s \Delta S h^4}{2L_s v k_T} = \frac{g \beta_s \Delta S h^3}{2L_s}.$$ 

Therefore,

$$Ra_{\text{Ref}0} = Ra_T \left( \frac{h}{H} \right)^3 \frac{h}{2L_T} \quad \text{and} \quad Ra_{s0} = Ra_s \left( \frac{h}{H} \right)^3 \frac{h}{2L_s},$$

where $T_{Z0}$ and $S_{Z0}$ are the instantaneous temperature and salinity gradient at the interface; $Ra_T$ and $Ra_s$ are our input thermal and salinity Rayleigh numbers based on total height and temperature and salinity jump across the interface; now substituting these back into the condition for onset of convection we get

$$Ra_{\text{Ref}0} = \frac{Ra_{\text{Ref}}}{\tau} \approx Ra_{cr},$$ 

$$Ra_T \left( \frac{h}{H} \right)^3 \frac{h}{2L_T} \left( \frac{1}{\tau k_T} \left( \frac{1}{2} \right) \right) \approx Ra_{cr},$$

$$\left( \frac{h}{H} \right)^3 = Ra_T \left( \frac{2L_s}{h} \right) \left( \frac{1}{\tau k_T} \left( \frac{1}{2} \right) \right),$$

$$\left( \frac{h}{H} \right)^3 = \left( \frac{Ra_T}{2L_s} \right) \left( \frac{1}{\tau k_T} \left( \frac{1}{2} \right) \right)^{1/3} \left( \frac{2L_s}{h} \right)^{1/3}.$$ 

The ratio $(h/2L_s)$ indicates the relative position of density extreme with respect to salinity diffusion length. This can be obtained by differentiating the net density profile due to both temperature and salinity and equating it to zero; the expression is $h/2L_s = \ln(1/R_{\rho})/(1-\tau)$ and hence we can write the final expression for the unstable layer thickness as

$$\left( \frac{h}{H} \right)^3 \approx \left( \frac{Ra_T}{2L_s} \right) \left( \frac{1}{\tau k_T} \left( \frac{1}{2} \right) \right)^{1/3} \left( \frac{2L_s}{h} \right)^{1/3}.$$ 

The analysis of Matthews\textsuperscript{55} indicates that for a system with unstable density profile bounded at the top and bottom by stable density stratification, the predominant wavelength of unstable waves that determines the width ($\lambda$) of the convection cell will be proportional to the unstable layer height [with a proportionality constant which is about 2 (Ref. 55)].

Hence in the above expression we can replace $(h/H)$ with $(\lambda/H)$ with a constant of proportionality (1.7 is the best value in our case) as presented in

$$\frac{\lambda}{H} = 1.7 \left( \frac{657 \left[ 1 - R_{\rho}^{3/2} \right]}{Ra_T \tau k_T} \right)^{1/3} \left( \frac{1}{R_{\rho} \sqrt{\tau}} \right)^{-1/6}.$$ 

Note that the Eq. (3.1) for the finger width has a singularity when $R_{\rho}=1/\sqrt{\tau}$. This singularity occurs because as the density stability ratio is increased, the net density profile at the interface switches from unstable to stable. Hence for $R_{\rho}$ values greater than or equal to $1/\sqrt{\tau}$ the density profile does not have regions of unstable density stratification. When the density profile becomes stable our arguments for onset of instability presented here are not applicable. For the values $R_{\rho}$ greater than 1 and lower than 10 (for the thermohaline system), expression (3.1) can be used to obtain the width of the finger. In Fig. 9, we have compared the values of the finger width from the above analysis with the incipient finger widths obtained from the numerical simulation. In the case of the simulation with $R_{\rho}=10$, we estimate the finger width by taking $R_{\rho}=9$ [to avoid the singularity in Eq. (3.1)]. The agreement between the finger width estimated from Eq. (3.1) and the numerical simulation is remarkably close. We have used very simple arguments with proportionality constant of 1.7 and $Ra_{cr}$ value of 657. The dashed line in the figure indicates $1.7 \times (Ra_T/657)^{1/3}$ fit, which to the first order captures the trend in variation of the nondimensional finger width. The nondimensional finger width varies as the cube root of Rayleigh number and this clearly indicates that the dimensional finger width is independent of the height of the domain and depends only on the temperature and salinity jump across the interface.
2. Velocity scale

We consider next the rate of change of potential energy and rate of dissipation to estimate the velocity in the finger-convection cells. At quasi-steady-state, the rate of change of potential energy must balance viscous dissipation,

\[ \text{rate of change of potential energy per unit volume} \sim \Delta \rho g W, \]

rate of viscous dissipation of kinetic energy per unit volume \( \sim \mu \left( \frac{2W}{\lambda} \right)^2 \),

where \( \Delta \rho \) is the density difference between the fluid in the finger (\( \rho_f \)) and the horizontal average density at the same level (\( \rho_{av} \)), \( W \) is the dimensional vertical velocity in the finger, \( \mu \) is the fluid viscosity, and \( \lambda \) is the finger width. Equating them, we get

\[
\frac{\Delta \rho}{\rho} g W = \frac{\mu}{\rho} \left( \frac{2W}{\lambda} \right)^2, \tag{3.2a}
\]

\[
W = \frac{\Delta \rho}{\rho} \frac{g}{\mu} \left( \frac{\lambda^2}{4} \right). \tag{3.2b}
\]

The density difference \( (\Delta \rho) \) between the finger fluid and the ambient fluid at that level is \( (\rho_f - \rho_{av}) \), which depends on the initial temperature and salinity differences across the interface and also on how diffusion has modified these contrasts as they flow in the opposite direction. As salt diffuses much more slowly than the heat, the salinity contrast is not affected in comparison to the temperature contrast as shown before. Hence by assuming that all the salinity in the fingers is retained and a fraction of the original thermal contrast is present, we can express \( \Delta \rho/\rho \) in terms of \( R_\rho \) as follows:

\[
\frac{\Delta \rho}{\rho} = \frac{\rho_f - \rho_{av}}{\rho} = (C_1 \beta_T \Delta T) = \beta_T \Delta T \left( \frac{1 - R_\rho C_1}{R_\rho} \right), \tag{3.2c}
\]

where \( C_1 \) is a fraction that represents temperature contrast that is remaining. We nondimensionalize vertical velocity \( (W) \) and finger width \( (\lambda) \) using \( k_T/H \) and \( H \), respectively. Equation (3.2a) becomes

\[
\frac{W}{(k_T/H)} \sim \frac{g}{(k_T/H)} \beta_T \Delta T \left( \frac{1 - R_\rho C_1}{R_\rho} \right) \frac{\lambda^2 H^2}{4 \nu H^2} = \frac{Ra_T 1/3}{R_\rho}. \tag{3.3}
\]

In Table II, we present maximum, horizontally averaged, absolute value of vertical velocity observed in the present simulation for all the cases. In Fig. 10, we compare those vertical velocities with the estimated value of vertical velocity obtained from Eq. (3.3). The plot indicates that there is a close agreement between the estimate from scale analysis and the values obtained from numerical simulation. Vertical velocity decreases with increase in density stability ratio \( (R_\rho) \) and is proportional to the 1/3 power of thermal Rayleigh number.

3. Heat and salt fluxes

Turner proposed a flux law which states that the convective salinity flux \( F_S = \langle W'S' \rangle \) should scale as

\[
F_S \propto \Delta S^{a/3}, \tag{3.4}
\]

where \( a = 4/3 \). Equation (3.4) is known as the “four-thirds” law and it assumes that the ratio of convective flux to the conduction flux is a function of Rayleigh number only. In the case of single-scalar convection, Eq. (3.4) is in fair agree-
ment with the experimental observations. For finger convection many authors have found reasonable agreement with the four-thirds law while others observed departure from the 4/3 power. Previous numerical simulations (e.g., Refs. 56–58) and laboratory experiments in heat-salt system20 have shown that the four-thirds law is good for predicting salinity flux. However, McDougall and Taylor,21 who extended the laboratory experiment to lower values of $R_f$, argued that $a = 1.23$ at low values of $R_f$. The variation of salinity flux with salinity Rayleigh numbers obtained in our simulation is presented in Fig. 11. Over a wide range of $R_{S}$ covered in the simulations (six decades), salinity flux correlates well with $R_{S}^{4/3}$ and supports the 4/3 law [Eq. (3.4)] proposed by Turner.18

The other quantity of direct interest to oceanographers is the buoyancy flux ratio ($R_f$). It is the ratio of potential energies transported vertically by finger-convection cells due to the temperature and salinity contrasts. As the thermal diffusion is about 100 times larger than salt diffusion, a part of the potential energy transported by the temperature field is lost due to the horizontal diffusion of heat between the upward moving and downward moving fingers.

One more factor to keep in mind while reporting flux ratio is the quasisteady nature of finger evolution process. Recently, Worster59 found that the double-diffusive interfaces are intrinsically time dependent. He concluded that fluxes can depend significantly on initial conditions and the history of evolution. This observation is significant in the context of our results. We have observed two regimes in finger dynamics during the evolution of the fingers. In the first regime for high Rayleigh number simulations, thin fingers evolve at the interface, and then they grow with time. During the evolution, neighboring fingers merge to form wider fingers. Mixed layers form at the top and bottom boundaries of the domain. Convection in the mixed layers limits further growth of the fingers and length remains constant for some time. Slowly the convection dies in the top and bottom regions of the domain and finger length starts increasing again. Fingers slowly fade away after a long time. In the second regime, simulation for low Rayleigh numbers, wide fingers evolve, and they slowly grow with time without merging and ultimately touch the system boundaries and no large-scale convection is observed in the mixed layer. The first process described above is typical in the laboratory experiments. The dynamics of this kind is often assumed to be of quasisteady state. The dynamics of the second kind is always time dependent. Nevertheless, both the dynamics of double-diffusive processes depend on the initial condition of the system as indicated by Worster.59 Hence, to bring about the uniformity in calculating the interfacial heat and salt fluxes in both high and low Rayleigh number simulations, we have considered the finger system between the time when fingers start evolving at the interface to the time when they grow and occupy 30%–40% height of the computational domain. The fluxes are time averaged within this period for all the 24 cases. The transient nature of the flux variation is presented in Fig. 12(a); in this case $R_{T}=7 \times 10^6$ and $R_p=10$. Two vertical lines in Fig. 12(a) indicate the time period over which flux ratio ($R_f$) is averaged. It is to be noted that Pringle et al.60 observed similar variation of mass flux as a function of time in the laboratory experiments using Hele-Shaw cell.

FIG. 11. Variation of salinity flux ($-\beta S F_{S}$) with salinity Rayleigh number ($R_{S}$). The slope of the fitted line is 1.52.

FIG. 12. (Color online) (a) Transient variation of $R_f$ in one simulation for which $R_p=10$ and $R_{T}=7 \times 10^6$; time period within which $R_f$ is averaged is marked by two vertical lines; (b) variation of $R_f$ with thermal Rayleigh number $R_{T}$ and density stability ratio $R_p$. The value of $R_f$ asymptotes at high $R_{T}$ for the ranges of $R_p$ considered in the analysis. The plot clearly demonstrates that $R_f$ depends on both $R_p$ and $R_{T}$, the effect of $R_{T}$ being more pronounced than $R_p$.
The variation of time averaged flux ratio as a function of $Ra_f$ for different density stability ratios ($R_f$) is presented in Fig. 12(b). From the figure, it is evident that $R_f$ has a systemic dependence on both Rayleigh number and density stability ratio. As the Rayleigh number is increased, for a given value of $R_p$, initially $R_f$ increases sharply and then attains a constant value. Other investigators in laboratory experiments and numerical simulations have documented this trend of $R_f$ increases with the decreases in $R_p$. In the present study, at high Rayleigh number ($7 \times 10^6 \leq Ra_f \leq 7 \times 10^9$), flux ratio has an average value of 0.64 for $R_p$ ranging from 1.5 to 10. Earlier numerical simulations for two-layer system at high Rayleigh numbers $^{29,57}$ and experiments of Turner $^{18}$ have reported the values of $R_f$, which are around 0.6, which is in good agreement with our study. At low $Ra_f=7 \times 10^3$, the value of $R_f$ is around 0.185 ($\pm 0.055$) for the range of $R_p$ from 1.5 to 10. Thus present results demonstrate that the flux ratio $R_f$ depends strongly on Rayleigh number apart from density stability ratio, with the tendency of it approaching 1 at high Rayleigh numbers. All the results presented here clearly indicate that the characteristics of finger convection, whether it is finger width ($\lambda$) or velocity ($w$) or flux ratio ($R_f$), cannot be described with a single parameter $R_p$.

We have shown that the flux ratio $R_f$ depends on Rayleigh number and density stability ratio $R_p$. Next we relate flux ratio with velocity and width of the finger using counterflow heat-exchanger analogy. Figure 13(a) depicts a few convection cells between two layers of fluids having salinity and temperature contrasts, forming a double-diffusive system in the finger regime. Figure 13(b) indicates one of the finger cells, in which hot saline fluid is flowing from top to bottom. As the fluid in this cell flows down, it loses heat to the neighboring fingers carrying cold fresh fluid upward$^{34}$ and the system is similar to a counterflow heat exchanger. Salt also diffuses to the neighboring fingers; however, due to the lower diffusivity of salt, horizontal salinity flux is negligible, which is evident from Fig. 4(a). The flux ratio $R_f$ is the nondimensional representation of the ratio of net transport of heat to the net transport of salt by the finger cell. With the low diffusivity ratio of the thermohaline system, salt transport occurs with little horizontal diffusion and in the best situation the vertical heat transport can at most match salt transport and hence the maximum value of $R_f$ will be 1. Hence the flux ratio approaches unity when the horizontal heat loss tends to zero. Therefore $R_f$ can be represented as follows:

$$R_f = \frac{(\beta_f W'T')_{\text{actual}}}{(\beta_f W'S')_{\text{ideal}}} = \frac{(\beta_f W'T')_{\text{ideal}} - Q_{\text{loss}}}{\beta_f W'S'} = \left[1 - \frac{Q_{\text{loss}}}{(\beta_f W'T')_{\text{ideal}}}\right].$$  \hspace{1cm} (3.5)

Hence, the variations observed in the $R_f$ can be attributed to the changes in $Q_{\text{loss}}$ ($\beta_f q/pC_p$; $C_p$ is the specific heat capacity and $q$ is the horizontal heat flux) due to the variations in finger velocity and the width of the finger. The second term on the right hand side of Eq. (3.5) is the nondimensional side heat loss in the finger and is equal to $(1-R_f)$. The parameters that control side heat loss are the width of the finger and the velocity.

In Fig. 14, we present the dependence of nondimensional side heat loss $(1-R_f)$ with finger width and velocity. Figures 14(a) and 14(b) indicate that side heat loss in the finger is inversely proportional to the velocity and is proportional to the finger width. In Fig. 14(c), we have plotted the variation of $(1-R_f)$ against the product of nondimensional velocity ($w$) and finger aspect ratio ($H/\lambda$). Data collapse to a single curve over a wide range of $Ra_f$ and $R_p$ covered in our simu-
lation. A higher velocity means smaller residence time and hence lower side heat loss in comparison to the vertical advection. However, the dependence on nondimensional finger width is counterintuitive. As the velocity and the finger widths are inversely related, the above argument again suggests that narrow fingers would have higher velocity and hence lower side heat loss. Thus the variation of $R_f$ observed is due to variation in the side heat loss in the finger-convection cell, which in turn depends on finger width and finger velocity. The velocity in the finger chiefly determines the time available for the lateral diffusion of heat more than salinity and controls the flux ratio of the finger-convection system. Results presented here also indicate that wide fingers will have lower flux ratios compared to the narrow fingers, which is in contradiction to assumptions in earlier theories.

Finally we discuss the low values of $R_f$ reported by McDougall and Taylor in laboratory experiments of heat-salt system at low values of density stability ratios. Some of the salient features of their experiments can be summarized as follows:

1. The buoyancy flux ratio $R_f$ was found to remain significantly lower than 1, even though $R_f$ in their experiments was as low as 1.2, which is of interest in oceanographic application.
2. The salinity steps ($\Delta S$) were the smallest achieved so far in laboratory experiments.
3. Results for the buoyancy flux ratio $R_f$ indicate dependence on $\Delta S$, with small value of $\Delta S$ resulting in a small value of flux ratio ($R_f$).

A careful interpretation of the above observations of McDougall and Taylor, in the light of our simulation results, explicitly suggests that the flux ratio $R_f$ depends on the magnitude of the salinity step $\Delta S$ (hence Rayleigh number) rather than $R_p$ alone. They measured the values of fluxes at effectively low Rayleigh numbers that resulted in different values of $R_f$. They, however, have not observed the significant variations in finger width and velocity as the ranges of Rayleigh numbers covered in their experiments were not large enough to see any noticeable changes in the finger structures.

IV. CONCLUSIONS

A series of 24 numerical simulations was performed in a two-layer, heat-salt system, covering a wide range of density stability ratios and Rayleigh numbers. Our results indicate that finger width is proportional to $Ra_T^{-1/3}$, which is quite different from the case of linearly stratified systems for which width is proportional to $Ra_T^{-1/4}$. We corroborated this result through scale analysis. Scale analysis was also used to find how the width of the finger and the velocity depend on Rayleigh number and density stability ratio. The vertical velocity in the fingers was shown to be directly proportional to $Ra_T^{1/3}$ and inversely proportional to $R_p$. The agreement between the scale analysis and results from numerical simulations is reasonably good. We found that the wide fingers have low velocity and lower values of flux ratios compared to the narrow fingers. This finding is contrary to the prevalent notion that assumes higher values of flux ratios and velocities for thick fingers compared to thin fingers.

The results obtained from the simulations indicate that the flux ratio is a function of both $R_p$ and $Ra_T$. With this insight, one can explain the large variations of $R_f$ reported in the literature (refer Fig. 2). The data reported in the literature have been obtained under various conditions. Some have been from laboratory experiments and others are from field observations. In all these cases salinity and temperature steps between the two mixed layers at the interface are different.
and hence the corresponding Rayleigh numbers are not the same. As shown in Fig. 1(a), even in a single vertical temperature trace, there are mixed layers of varying thickness with different values of temperature and salinity steps across them. These variations contribute to the variations in Rayleigh numbers and hence the observed flux ratio.

The results of our simulation suggest that $R_f$ is not the only parameter that controls the vertical transport of fluxes by fingers. At high Rayleigh numbers, where the stability of the system is primarily governed by the parameter $R_f$, $R_f$ shows little dependence on Rayleigh number [see Fig. 12(b)] and hence the parameter $R_f$ is sufficient to explain the variation of fluxes in the high Rayleigh number regime. However, as the Rayleigh number decreases (at a fixed value of $R_f$) the system changes from being advection dominated to diffusion dominated (Fig. 3). The overall result of this transition is a low value of $R_f$ and this is depicted in Fig. 12(b). The flux at low Rayleigh numbers depends on both $R_f$ and $R_{R_f}$ (or $R_{A_f}$). Hence, at least two of the three governing parameters $R_p$, $R_{T_f}$, and $R_{A_f}$ are needed to predict characteristics of finger convection and to quantify the fluxes transported by the fingers.

A counterflow heat-exchanger analogy is used for understanding dependence of $R_f$ on finger width and velocity. As it is evident from Fig. 3, the lower value of $R_f$ observed for low Rayleigh number situation is due to the significant horizontal diffusion of heat.

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49 See EPAPS Document No. E-PHFLE6-21-036901 for code validation and grid dependence tests. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html.